Classical Electrodynamics of a Particle with Maximal Acceleration Corrections*

A. Feoli[a†] G. Lambiase[a‡] G. Papini[b§] G. Scarpetta[a,c]

[a] *Dipartimento di Fisica Teorica e s.m.s.a. Università di Salerno, 84081 Baronissi (SA), Italy*

[a] *Istituto Nazionale di Fisica Nucleare, Sezione di Napoli*

[b] *Department of Physics, University of Regina, Regina, Sask. S4S 0A2, Canada*

[c] *International Institute for Advanced Scientific Studies Vietri sul Mare (SA), Italy*

Abstract

We calculate the first order maximal acceleration corrections to the classical electrodynamics of a particle in external electromagnetic fields. These include additional dissipation terms, the presence of a critical electric field, a correction to the cyclotron frequency of an electron in a constant magnetic field and the power radiated by the particle. The electric effects are sizeble at the fields that are considered attainable with ultrashort TW laser pulses on plasmas.

PACS: 41.10

*Research supported by MURST fund 40% and 60%, DPR 382/80

† *E-mail*: Feoli@vaxsa.csied.unisa.it

‡ *E-mail*: lambiase@vaxsa.csied.unisa.it

§ *E-mail*: papini@cas.uregina.ca

1 Introduction

In an attempt to incorporate quantum aspects in the geometry of space-time, Caianiello wrote a series of papers about a theory called "Quantum Geometry" [1], where the position and momentum operators are represented as covariant derivatives and the quantization is interpreted as curvature of phase space. A novel feature of the model is the existence of a maximal acceleration [2]. Limiting values for the acceleration were also derived by several authors on different grounds and applied to many branches of physics such as string theory, cosmology, quantum field theory, black hole physics, etc. [3].

The model proposed by Caianiello and his coworkers to include the effects of a maximal acceleration in a particle dynamics consisted in enlarging the space-time manifold to an eight-dimensional space-time tangent bundle TM. In this way the invariant line element becomes

$$d\tilde{\tau}^2 = g_{AB} dX^A dX^B, \quad A, B = 1, \ldots, 8 \tag{1.1}$$

where the coordinates of TM are

$$X^A = \left(x^\mu; \frac{c}{A} \frac{dx^\mu}{d\tau} \right), \quad \mu = 1, \ldots, 4 \tag{1.2}$$

$A = mc^3/\hbar$, and

$$g_{AB} = g_{\mu\nu} \otimes g_{\mu\nu} \,. \tag{1.3}$$

The metric (1.1) can be written as

$$d\tilde{\tau}^2 = g_{\mu\nu} (dx^\mu dx^\nu + \frac{c^2}{A^2} d\dot{x}^\mu d\dot{x}^\nu) \,, \tag{1.4}$$

and an embedding procedure can be developed [4] in order to find the effective space-time geometry in which a particle moves when the constraint of a maximal acceleration is present. In fact, if we find the parametric equations that relate the velocity field $\dot{x}^\mu$ to the first four coordinates x^μ, we can calculate the effective four dimensional metric $\tilde{g}_{\mu\nu}$ on the hypersurface locally embedded in TM.

In Section 2 we derive the quantum corrections to the metric for a charged particle moving in an electromagnetic field and the corresponding first order corrections to the equations of motion. In Section 3 we analyze some consequences of the corrections for particular instances. We derive the radiated power in Section 4. Section 5 contains a discussion.

2 Effective Geometry

Consider the Minkowski metric $\eta_{\mu\nu}$ (of signature -2) as the background metric in Eq. (1.4). A particle of mass m and charge q moves in this background under the action of an externally applied electromagnetic field. The classical equation of motion of the particle is

$$d\dot{x}^{\mu} = -\frac{q}{mc} F^{\mu}{}_{\nu} dx^{\nu} \tag{2.1}$$

where $F_{\mu\nu}$ is the classical electromagnetic field tensor. Eq. (2.1) may be taken as the first order approximation to the real velocity field of the particle. If we substitute Eq. (2.1) into (1.4) we can calculate the correction of order A^{-2} to the classical background metric. We can iterate this procedure, by calculating the new velocity field and substituting it again into the metric to obtain the correction A^{-4}, and so on. However the value of the maximal acceleration is very high and we can neglect the $O(1/A^4)$ terms. This leads to the new metric

$$d\tilde{\tau}^2 \simeq \left(\eta_{\mu\nu} - \frac{q^2}{A^2 m^2} F_{\mu\lambda} F^{\lambda}{}_{\nu} \right) dx^{\mu} dx^{\nu} \equiv \tilde{g}_{\mu\nu} dx^{\mu} dx^{\nu} \,. \tag{2.2}$$

The effective geometry is curved by the acceleration due to the interaction of the charged particle with the electromagnetic field and this curvature affects the motion of the particle itself.

The modified equation of motion of the charged particle may be obtained from the action

$$S = \int (-mc^2 \sqrt{\tilde{g}_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}} + q A_{\mu} \dot{x}^{\mu}) d\tau \tag{2.3}$$

and leads to the equations

$$\ddot{x}^{\sigma} + \Gamma^{\sigma}{}_{\alpha\beta} \dot{x}^{\alpha} \dot{x}^{\beta} + \frac{q}{mc} \tilde{g}^{\sigma\rho} F_{\rho\gamma} \dot{x}^{\gamma} = 0 \,, \tag{2.4}$$

where

$$\Gamma^{\sigma}{}_{\alpha\beta} = \frac{q^2}{2A^2 m^2} \tilde{g}^{\sigma\nu} [F^{\lambda}{}_{\nu} (F_{\lambda\alpha,\beta} + F_{\lambda\beta,\alpha}) + F^{\lambda}{}_{\alpha} F_{\beta\nu,\lambda} + F^{\lambda}{}_{\beta} F_{\alpha\nu,\lambda}] \tag{2.5}$$

is the connection, and the dot means derivation with respect to $\tilde{\tau}$.

The contravariant components of the metric tensor $\tilde{g}^{\sigma\nu}$ then follow from the equation $\tilde{g}^{\sigma\nu}\tilde{g}_{\nu\rho} = \delta^{\sigma}_{\rho}$ in the approximation $\frac{q^2}{A^2m^2}F^{\sigma}_{\ \lambda}F^{\lambda\nu} \ll \eta^{\sigma\nu}$ and are

$$\tilde{g}^{\sigma\nu} \simeq \eta^{\sigma\nu} + \frac{q^2}{A^2m^2}F^{\sigma}_{\ \lambda}F^{\lambda\nu}\,. \tag{2.6}$$

Hereafter the raising and lowering of the indices is made with the tensor $\eta_{\mu\nu}$. Obviously the corrections disappear in the limit $A \to \infty$ and we find the classical equation of motion.

To analyze the physical consequences of this correction, it is useful to write the equation of motion (2.4) in terms of the four-momentum $p^\mu = mc\dot{x}^\mu$ and the four-velocity $dx^\mu/d\tilde{\tau} = u^\mu = (cdt/d\tilde{\tau}, \vec{v}dt/d\tilde{\tau})$

$$\frac{dp^\mu}{d\tilde{\tau}} = \frac{q}{c}\tilde{g}^{\mu\nu}F_{\nu\rho}u^\rho - \frac{q^2}{2A^2m}\tilde{g}^{\mu\nu}(F^{\lambda}_{\ \nu}F_{\lambda\alpha,\beta} + F^{\lambda}_{\ \alpha}F_{\beta\nu,\lambda})u^\alpha u^\beta\,. \tag{2.7}$$

3 First Order Corrections

The particle embedded in the external field experiences a curved manifold even if the background metric is flat. This curvature affects the motion of the particle and the rate of change of its energy $\varepsilon \equiv p^0$ in time. We analyze first the simple case when only an electric field $\vec{E}$ is present. We obtain from (2.7)

$$\frac{d\varepsilon}{dt} = q\vec{E}\cdot\vec{v}\left(1 + \frac{q^2|\vec{E}|^2}{A^2m^2}\right) + \frac{q^2c^3}{A^2m\frac{d\tilde{\tau}}{dt}}\left[\frac{1}{2}\frac{d|\vec{E}|^2}{dt} + \frac{1}{c^2}\left(\vec{v}\cdot\frac{\partial\vec{E}}{\partial t}\right)\vec{E}\cdot\vec{v} + v^jE_i\partial_iE_j\right]\,. \tag{3.1}$$

The difference of (3.1) from the usual classical equation becomes evident when $\vec{E}\cdot\vec{v} = 0$. In this case we find

$$\frac{d\varepsilon}{dt} = \frac{q^2c^3}{2A^2m\frac{d\tilde{\tau}}{dt}}\left[\frac{d|\vec{E}|^2}{dt} + 2v^jE_i\partial_iE_j\right]\,, \tag{3.2}$$

which is an additional dissipation term. On the other hand the orthogonality condition of $\vec{E}$ and $\vec{v}$ cannot be imposed in general because of the perturbing effect of the acceleration on the motion itself.

The quantity $\frac{d\tilde{\tau}}{dt}$ in Eq. (3.1) is defined by

$$\frac{d\tilde{\tau}}{dt} = c\sqrt{1 - \frac{v^2}{c^2} - \frac{q^2|\vec{E}|^2}{A^2m^2}}\,. \tag{3.3}$$

Since $\frac{d\tilde{\tau}}{dt}$ must be real, we have

$$|\vec{E}| < \frac{mA}{|q|}\sqrt{1 - \frac{v^2}{c^2}}\,, \tag{3.4}$$

i.e.,the intensity $|\vec{E}|$ is limited by the critical electric field

$$E_c = \frac{mA}{|q|}\sqrt{1 - \frac{v^2}{c^2}}\,. \tag{3.5}$$

In the particle's rest frame E_c becomes

$$E_c = \frac{mA}{|q|}\,. \tag{3.6}$$

If the particle is an electron, we find $E_c(m_{e^-}) \simeq 2.1 \cdot 10^{16} N/C$ when the maximal acceleration is mass-dependent, and $E_c \simeq 5.6 \cdot 10^{39} N/C$ when A refers to the Planck mass. Similarly for the proton we find: $E_c(m_{p^+}) \simeq 10^{24} N/C$ and $E_c \simeq 10^{43} N/C$ respectively.

It is interesting to observe that a similar critical value for the electric field was found for the open bosonic string propagating in an external electromagnetic field. In this case $E < T/|q_a|$ $(a = 1, 2)$ where T is the string tension and q_a are the charges at the ends of the string [5]. This condition is satisfied automatically when the background electromagnetic field is described by the Born-Infeld Lagrangian rather then by the Maxwell Lagrangian [6]. It is therefore possible to expect a connection between the Born-Infeld and the Caianiello Lagrangians [7]. The relation between acceleration and E_c was also analyzed by Gasperini [8]

It is also instructive to calculate Eq. (2.7) explicitly in the case of a particle with charge $q = -e$, moving in a constant electromagnetic field of components $\vec{E} = (E_x, E_y, 0)$ and $\vec{B} = (0, 0, B)$. We find

$$\frac{d\varepsilon}{dt} = -e(E_x\dot{x} + E_y\dot{y}) + \frac{e^3}{A^2m^2}(E_x\dot{x} + E_y\dot{y})(|\vec{E}|^2 - B^2)\,. \tag{3.7}$$

In this case all dissipation terms due to the maximal acceleration disappear only if $\vec{E} = 0$.

Finally, we consider the problem of charged particles moving in a constant magnetic field of components $\vec{B} = (0, 0, B)$. From equation (2.7), the equations of motion for a circular path are

$$\begin{cases} x(t) = r_0 \sin \omega t \\ y(t) = r_0 \cos \omega t \end{cases} , \tag{3.8}$$

where

$$\omega = \frac{eB}{m} + \frac{e^3 B^3 c^2}{A^2 m^3} = \omega_c + \triangle\omega , \tag{3.9}$$

and $\triangle\omega = \omega_c^3 c^2 / A^2$ is the first correction to the classical frequency ω_c. This is the same correction of Ref. [9] if the "rigidity" correction β-term there obtained is calculated for $v = 0$. Due to the existence of higher derivative terms in the Lagrangian, the corrections to the cyclotron frequency found in [9] depend in fact on the velocity of the particle. From equation (3.9) we obtain

$$\frac{\triangle\omega}{\omega_c} = \left(\frac{eBc}{Am}\right)^2 . \tag{3.10}$$

The limit coming from the (g-2)/2 experiment for non relativistic electrons trapped in a magnetic field $B \sim 50KG$ is [10]

$$\frac{\triangle\omega}{\omega_c} \leq 2 \cdot 10^{-10} . \tag{3.11}$$

From (3.10) we find $\triangle\omega/\omega_c \sim 6 \cdot 10^{-15}$ when the maximal acceleration is mass-dependent and $\triangle\omega/\omega_c \sim 2 \cdot 10^{-60}$ when the maximal acceleration is considered a universal constant. Both results are in agreement with the experimental upper bound (3.11).

4 Power Radiated by a Charged Particle

The total four-momentum radiated by a charge moving in an electromagnetic field is given by the well-known formula [11]

$$\Delta P^\mu = -\frac{2e^2}{3c} \int |\ddot{x}|^2 dx^\mu . \tag{4.1}$$

If the contributions of the gradients of the electromagnetic fields can be neglected in (2.5), which is certainly legitimate at the highest gradients available at present or in a foreseable future, one finds $\Gamma^{\sigma}_{\alpha\beta} \approx 0$, and (2.4) becomes

$$\ddot{x}^{\sigma} = \frac{e}{mc}\tilde{g}^{\sigma\rho}F_{\rho\gamma}\dot{x}^{\gamma} \approx \frac{e}{mc}\left(\eta^{\sigma\rho} + \frac{e^2}{m^2A^2}F^{\sigma}_{\lambda}F^{\lambda\rho}\right)F_{\rho\gamma}\dot{x}^{\gamma}\,. \tag{4.2}$$

Substituting (4.2) into (4.1) we obtain

$$\Delta P^{\mu} = \Delta P^{\mu}_{(0)} + \Delta P^{\mu}_{(A)}\,, \tag{4.3}$$

where

$$\Delta P^{\mu}_{(0)} = -\frac{2e^4}{3m^2c^5}\int (F^{\gamma}{}_{\beta}\dot{x}^{\beta})(F_{\gamma\alpha}\dot{x}^{\alpha})dx^{\mu} \tag{4.4}$$

is the classical result, and

$$\Delta P^{\mu}_{(A)} = -\frac{2e^4}{3m^2c^5}\frac{e^2}{A^2m^2}\int (F_{\sigma\gamma}\dot{x}^{\gamma})F^{\sigma}_{\lambda}F^{\lambda\beta}(F_{\beta\alpha}\dot{x}^{\alpha})dx^{\mu} \tag{4.5}$$

represents the maximal acceleration contribution. When $\vec{B} = 0$ and $\vec{E}$ is parallel to $\vec{\beta} = \vec{v}/c$, Eq. (4.5) for $\mu = 0$ becomes

$$\frac{d\varepsilon(A)}{dx^0} = \frac{2e^4}{3m^2c^3}\frac{e^2}{A^2m^2}|\vec{E}|^4\,, \tag{4.6}$$

and (4.4) yields

$$\frac{d\varepsilon}{dx^0} = \frac{2e^4}{3m^2c^3}|\vec{E}|^2\,. \tag{4.7}$$

The ratio of the classical dissipation term (4.7) to the maximal acceleration term (4.6) for electrons is

$$\Gamma = \frac{d\varepsilon/dx^0}{d\varepsilon(A)} = \frac{m^2A^2}{e^2|\vec{E}|^2} \sim 1.21\,\frac{10^{36}}{|\vec{E}|^2}\,, \tag{4.8}$$

while the total dissipated power is

$$P_{tot} = \frac{2e^4}{3m^2c^3}|\vec{E}|^2\left(1 + \frac{e^2|\vec{E}|^2}{m^2A^2}\right)\,. \tag{4.9}$$

5 Conclusions

Even at their lowest order the maximal acceleration corrections to the classical electrodynamics of a particle offer interesting aspects. These include the existence of a critical field beyond which approximation and equations break down and corrections to the cyclotron frequency of an electron in a constant magnetic field. While the latter effect is compatible with present stringent experimental upper limits from the $(g-2)/2$ experiment, it may by its very size be difficult to observe directly. On the contrary, the field $E_c \sim 2.1 \cdot 10^{16}$N/C for electrons is very close to values $E \sim 5 \cdot 10^{15}$N/C that are presently considered as attainable by focusing TW laser beams in a plasma [12]. Radiation levels at these values of the electric field also become non-negligible as entailed by (4.6), (4.7) and (4.8). In fact the maximal acceleration contribution increases with the fourth power of $|\vec{E}|$ and contributes sizeable amounts to the power radiated at the highest field intensities. Increased radiation should of course be expected on intuitive grounds. This typical functional increase would constitute the signature of the process once the promise of high electric fields became a reality.

Acknowledgment

G.P. gladly acknowledges the continued research support of Dr. K. Denford, Dean of Science, University of Regina.

REFERENCES

1. E.R. Caianiello, *Lett. Nuovo Cimento* **25**, 225 (1979); **27**, 89 (1980); *Il Nuovo Cimento* **B59**, 350 (1980)

   E.R. Caianiello, *La rivista del Nuovo Cimento* **15** n.4 (1992) and references therein.

2. E.R. Caianiello, *Lett. Nuovo Cimento* **32**, 65 (1981)

   E.R. Caianiello, S. De Filippo, G. Marmo and G.Vilasi, *Lett. Nuovo Cimento* **34**, 112 (1982)

   E.R. Caianiello, *Lett. Nuovo Cimento* **41**, 370 (1984).

3. W.R. Wood, G. Papini and Y.Q. Cai, *Il Nuovo Cimento* **B104**, 361 and (errata corrige) 727 (1989)

   G. Papini, *Mathematica Japonica* **41**, 81 (1995)

   A. Das, *J. Math. Phys.* **21**, 1506 (1980)

   H.E. Brandt, *Lett. Nuovo Cimento* **38**, 522 (1983) and (errata corrige) **39**, 192 (1984); *Found. Phys. Lett.* **2**, 39 (1989) and references therein

   M. Toller, *Nuovo Cimento* **B102**, 261 (1988); *Int. J. Theor. Phys.* **29**, 963 (1990); *Phys. Lett.* **B256**, 215 (1991)

   B. Mashoon, *Physics Letters***A143**, 176 (1990) and references therein

   V.P. Frolov and N. Sanchez, *Nucl. Phys.* **B349**, 815 (1991)

   A.K. Pati, *Europhys. Lett.* **18**, 285 (1992)

   A. Feoli, *Nucl. Phys.* **B396**, 261 (1993)

   N. Sanchez, in *"Structure: from Physics to General Systems"* eds. M. Marinaro and G. Scarpetta (World Scientific, Singapore, 1993) vol. 1, pag. 118.

   S. Capozziello and A. Feoli, *Int. J. Mod. Phys.* **D2**, 79 (1993)

   E.R. Caianiello, S. Capozziello, R. de Ritis, A. Feoli, G. Scarpetta, *Int. J. Mod. Phys.* **D3**, 485 (1994)

4. E.R. Caianiello, A. Feoli, M. Gasperini and G. Scarpetta, *Int. Jour. Theor. Phys.* **29**, 131 (1990)

5. C.P. Burgess, *Nucl. Phys.* **B294**, 427 (1987)

   B.M. Barbashov and V.V. Nesterenko: *Introduction to the relativistic string theory* (World Scientific, Singapore,1990), pag. 96.

6. E.S. Fradkin and A.A. Tseytlin, *Nucl. Phys.* **B261**, 1 (1985); *Phys Lett.* **158B**, 516 (1985); **160B**, 69 (1985); **163B**, 123 (1985)

   A.A. Aboulsaood, C.G. Callan, C.R. Nappi and S.A. Yost, *Nucl. Phys.* **B280**, 599 (1987).

7. V.V. Nesterenko, A. Feoli and G. Scarpetta, *J. Math. Phys.* **36**, 5552 (1995).

8. M. Gasperini, *Gen.Rel. and Grav.* **24**, 219 (1992).

9. G. Fiorentini, M. Gasperini and G. Scarpetta, *Mod. Phys. Lett.* **A6**, 2033 (1991).

10. R.S.Van Dyck Jr, P.B.Schwinberg and H.G.Dehmelt, *Phys. Rev. Lett.* **59**, 26 (1987).

11. L.D. Landau and E.M. Lifshitz, The Classical Theory of Fields, Pergamon Press, Oxford, 1969.

12. D. Umstadter, J.K. Kim and E. Dodd, *Phys. Rev. Lett.* **76**, 2073 (1996).